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THE MEAN CURVATURE FOR p-PLANE

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Introduction

Let M be an n-dimensional Riemannian space. For the skew symmetric tensor $u_{\lambda_1 \cdots \lambda_p}$, $F_p(u)$ for $p = 1, \cdots, n$ are defined as follows:

$$F_1(u) = R_{\lambda\mu} u^\lambda u^\mu \ ,$$

$$F_p(u) = R_{\lambda\mu} u^{\lambda\rho_2 \cdots \rho_p} u^\mu{}_{\rho_2 \cdots \rho_p} + \frac{p-1}{2} R_{\lambda\mu\nu\omega} u^{\lambda\mu\rho_3 \cdots \rho_p} u^{\nu\omega}{}_{\rho_3 \cdots \rho_p} \ , \qquad p \geq 2 \ ,$$

where $R_{\lambda\mu\nu\omega}$ is the Riemannian curvature tensor and $R_{\lambda\mu} = R_{\alpha\lambda\mu}{}^\alpha$ is the Ricci tensor of M. Throughout this paper indices $\lambda, \mu, \nu, \cdots$ range from 1 to n, tensors and vectors will be represented with respect to the natural basis unless stated otherwise, and the summation convention is assumed for these indices. Concerning $F_p(u)$ the following theorems are known.

Theorem A [5, p. 64], [3]. *If the quadratic form $F_p(u)$ is positive definite in a compact Riemannian space, there exists no harmonic p-form other than the zero form.*

Theorem B [5, p. 67]. *If $F_p(u)$ is negative definite in a compact Riemannian space, there exists no Killing tensor field of degree p other than the zero tensor.*

Theorem C [4], [2]. *If $F_p(u)$ is negative definite in a compact Riemannian space for $p \leq n/2$, there exists no conformal Killing tensor field of degree p other than the zero tensor.*

In this paper in § 2 we shall give a geometric meaning of $F_p(u)$ in terms of the sectional curvature for a special form u to be called a simple form u, and § 3 is devoted to the discussion of the spaces in which $F_p(u)$ is independent of the simple form u.

1. Preliminaries

Let M be an n-dimensional Riemannian space. Consider a pair of orthonormal vectors $X = (X^\lambda)$ and $Y = (Y^\lambda)$ at a point $m \in M$. Then the sectional curvature of the 2-plane spanned by X and Y is given by

$$\rho(X, Y) = -R_{\lambda\mu\nu\omega} X^\lambda Y^\mu X^\nu Y^\omega \ .$$

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Let π be a p-plane at m. An orthonormal base $\{e_i\}$, $i = 1, \cdots, n$, is said to be adapted to π if $e_1, \cdots, e_p$ span π. Denote $e_i = \xi_i{}^\lambda \partial/\partial x^\lambda$, and define

$$\pi^{\lambda_1 \cdots \lambda_p} = \begin{vmatrix} \xi_1{}^{\lambda_1} \cdots \xi_p{}^{\lambda_1} \\ \cdots\cdots\cdots \\ \xi_1{}^{\lambda_p} \cdots \xi_p{}^{\lambda_p} \end{vmatrix} . \tag{1.1}$$

Consider another orthonormal base $\{e_i'\}$ adapted to π. Then

$$e_i' = \sum_{j=1}^{p} a_{ij} e_j , \qquad i = 1, \cdots, p ,$$

where the $p \times p$ matrix $A = (a_{ij})$ is orthogonal. Thus under the change of adapted bases we have

$$\pi'^{\lambda_1 \cdots \lambda_p} = \pm \pi^{\lambda_1 \cdots \lambda_p} , \tag{1.2}$$

and $\pi^{\lambda_1 \cdots \lambda_p}$ is determined for π within a sign. We shall call the tensor $\pi^{\lambda_1 \cdots \lambda_p}$ the simple p-vector of π, and denote it by π again. The ambiguity of signs does not matter in the following discussion.

2. The mean curvature for π

Let π be a p-plane at m, and $\{e_i\}$ an adapted base. Put

$$\rho(\pi_e) = \frac{1}{p(n-p)} \sum_{i=1}^{p} \sum_{j=p+1}^{n} \rho(e_i, e_j) ,$$

and prove that its value depends only on π. In fact, it will be seen as follows that $\rho(\pi_e)$ is independent of the choice of $\{e_i\}$.

Let $F_p(u)$ be the quadratic form of u defined in the introduction. Denote by $F_p(\pi_e)$ the $F_p(u)$ with $u^{\lambda_1 \cdots \lambda_p}$ to be $\pi^{\lambda_1 \cdots \lambda_p}$ of (1.1), and define

$$\bar{\rho}(\pi_e) = \frac{1}{p!(n-p)} F_p(\pi_e) ,$$

which is independent of the choice of adapted bases to π, because of (1.2). Thus for our purpose mentioned above it is sufficient to show that $\rho(\pi_e) = \bar{\rho}(\pi_e)$.

As $F_p(\pi_e)$ is a tensor equation, we may consider it written with respect to the adapted base $\{e_i\}$ of π. Then the components of e_i are $\delta_i{}^\lambda$, and the simple p-vector has the components

$$\pi^{\lambda_1 \cdots \lambda_p} = \begin{cases} \text{sign}\,(\lambda_1, \cdots, \lambda_p), & \text{if } (\lambda_1, \cdots, \lambda_p) \text{ is a permutation of } \{1, \cdots, p\}, \\ 0 & \text{other cases.} \end{cases}$$

Thus we have for $\lambda, \mu, \nu, \omega = 1, \cdots, p$

$$(2.1)\qquad \begin{aligned} \pi^{\lambda\rho_2\cdots\rho_p}\pi^{\mu}{}_{\rho_2\cdots\rho_p} &= (p-1)!\,\delta_{\lambda\mu}\,, \\ \pi^{\lambda\mu\rho_3\cdots\rho_p}\pi^{\nu\omega}{}_{\rho_3\cdots\rho_p} &= (p-2)!\,(\delta_{\lambda\nu}\delta_{\mu\omega}-\delta_{\lambda\omega}\delta_{\mu\nu})\,, \end{aligned}$$

and the following equations are valid:

$$R_{\lambda\mu}\pi^{\lambda\rho_2\cdots\rho_p}\pi^{\mu}{}_{\rho_2\cdots\rho_p} = (p-1)!\sum_{\lambda=1}^{p} R_{\lambda\lambda} = (p-1)!\sum_{\lambda=1}^{p}\sum_{\mu=1}^{n}\rho(e_\lambda, e_\mu)\,,$$

$$\begin{aligned} \frac{p-1}{2}R_{\lambda\mu\nu\omega}\pi^{\lambda\mu\rho_3\cdots\rho_p}\pi^{\nu\omega}{}_{\rho_3\cdots\rho_p} &= (p-1)!\sum_{\lambda,\mu=1}^{p} R_{\lambda\mu\lambda\mu} \\ &= -(p-1)!\sum_{\lambda=1}^{p}\sum_{\mu=1}^{p}\rho(e_\lambda, e_\mu)\,. \end{aligned}$$

Hence $\bar{\rho}(\pi_e) = \rho(\pi_e)$. Since $\rho(\pi_e)$ depends only on π, we denote it by $\rho(\pi)$ and call it the mean curvature for the p-plane π. We notice that the mean curvature for the p-plane spanned by $e_1, \cdots, e_p$ coincides with that for the $(n-p)$-plane spanned by $e_{p+1}, \cdots, e_n$.

3. A theorem analogous to Schur's theorem

In this section we shall determine the spaces in which $\rho(\pi)$ is independent of the p-plane π at each point. First we have

Lemma 1. *Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix whose diagonal elements are all zero. If A satisfies*

$$(3.1)\qquad \sum_{h,k=1}^{p} a_{i_h, i_k} = 0$$

for any $i_1 < \cdots < i_p$ taken from $\{1, \cdots, n\}$ and $n-1 > p > 1$, then A is the zero matrix.

Proof. For $\{i_1, \cdots, i_p\} = \{1, \cdots, p\}$ and $\{i_1, \cdots, i_p\} = \{2, \cdots, p+1\}$ from (3.1) we have, respectively,

$$(3.2)\qquad \sum_{i,j=1}^{p} a_{ij} = 0\,, \qquad \sum_{i,j=2}^{p+1} a_{ij} = 0\,,$$

which imply, due to $a_{ij} = a_{ji}$, that

$$\sum_{i=1}^{p+1} a_{i1} = \sum_{i=1}^{p+1} a_{i,p+1}\,.$$

Similarly,

$$\sum a_{i1} = \sum a_{i2} = \cdots = \sum a_{ip} = \sum a_{i,p+1}\,,$$

where $\sum$ denotes the summation over i from 1 to $p+1$. If we use $p+2$ instead of $p+1$, then

$$\sum' a_{i1} = \sum' a_{i2} = \cdots = \sum' a_{ip} = \sum' a_{i,p+2} ,$$

where $\sum'$ denotes the summation over i from 1 to p and $p+2$. Therefore we get

$$\begin{aligned} a_{p+1,1} - a_{p+2,1} &= a_{p+1,2} - a_{p+2,2} = \cdots = a_{p+1,p} - a_{p+2,p} \\ &= \sum a_{i,p+1} - \sum' a_{i,p+2} = \sum a_{p+1,i} - \sum' a_{p+2,i} . \end{aligned}$$

If we denote the above common value by k, then

$$pk = \sum_{i=1}^{p} (a_{p+1,i} - a_{p+2,i}) = \sum a_{p+1,i} - \sum' a_{p+2,i} = k ,$$

from which follows $k = 0$. Thus we have

$$a_{p+1,i} = a_{p+2,i} , \qquad i = 1, \cdots, p .$$

Similarly,

$$a_{p+1,i} = a_{p+2,i} = \cdots = a_{ni} , \qquad i = 1, \cdots, p .$$

The similar process for $\{1, \cdots, p-1, p+1\}$ leads us to

$$a_{pi} = a_{p+2,i} = \cdots = a_{ni} , \qquad i = 1, \cdots, p-1, p+1 ,$$

and finally we get

$$a_{ij} = k_j , \quad j = 1, \cdots, p; \quad i = 1, \cdots, n; \quad i \neq j .$$

As $a_{ij} = a_{ji}$, we obtain $a_{ij} = c$ for $i, j = 1, \cdots, p$, and hence $c = 0$ follows from (3.2). In a similar way, we know all the elements of A to be zero. q.e.d.

Now let us assume that $\rho(\pi)$ is independent of the p-plane at each point and takes the value $k/(n-p)$, where k is a scalar function. By the assumption we have $F_p(\pi_e) = p!\,k$, and hence

$$L_{\lambda\mu\nu\omega}\pi^{\lambda\mu\rho_3\cdots\rho_p}\pi^{\nu\omega}{}_{\rho_3\cdots\rho_p} = 0 \tag{3.3}$$

on taking account of (2.1), where

$$\begin{aligned} L_{\lambda\mu\nu\omega} = {} & (p-1)R_{\lambda\mu\nu\omega} - k(g_{\lambda\nu}g_{\mu\omega} - g_{\lambda\omega}g_{\mu\nu}) \\ & + \tfrac{1}{2}(R_{\lambda\nu}g_{\mu\omega} - R_{\lambda\omega}g_{\mu\nu} + R_{\mu\omega}g_{\lambda\nu} - R_{\mu\nu}g_{\lambda\omega}) . \end{aligned} \tag{3.4}$$

Now we may consider that (3.3) has been written with respect to the adapted base of (1.1). Then by virtue of (2.1) we get

$$\sum_{\lambda,\mu=1}^{p} L_{\lambda\mu\lambda\mu} = 0$$

for the base. Similarly, the analogous equations are valid for any p indices.

Thus, if we put $a_{\lambda\mu} = L_{\lambda\mu\lambda\mu}$, $(\lambda, \mu = 1, \cdots, n)$, then the $n \times n$ matrix $A = (a_{\lambda\mu})$ satisfies the condition of Lemma 1; consequently $a_{\lambda\mu} = 0$ follows.

On the other hand, we know [1, p. 196]

Lemma 2. *Let L be a tensor of type* (0,4) *satisfying*

$$L_{\lambda\mu\nu\omega} = -L_{\mu\lambda\nu\omega} = -L_{\lambda\mu\omega\nu}\ , \qquad L_{\lambda\mu\nu\omega} + L_{\mu\nu\lambda\omega} + L_{\nu\lambda\mu\omega} = 0\ .$$

If $L_{\lambda\mu\lambda\mu}$ for all λ and μ with respect to any orthonormal base are zero, then L is the zero tensor.

The tensor $L_{\lambda\mu\nu\omega}$ of (3.4) clearly satisfies the condition of Lemma 2. Thus we get

$$L_{\lambda\mu\nu\omega} = 0\ . \tag{3.5}$$

Transvecting $g^{\lambda\omega}$ with the last equation, we have

$$(2p - n)R_{\mu\nu} = [R - 2k(n-1)]g_{\mu\nu}\ , \tag{3.6}$$

where $R = g^{\lambda\omega}R_{\lambda\omega}$ is the scalar curvature. If $2p \neq n$, it follows that

$$k = \frac{n-p}{n(n-1)}R\ , \qquad R_{\mu\nu} = \frac{R}{n}g_{\mu\nu}\ ,$$

and substituting these values into (3.5) we get

$$R_{\lambda\mu\nu\omega} = \frac{R}{n(n-1)}(g_{\lambda\omega}g_{\mu\nu} - g_{\lambda\nu}g_{\mu\omega})\ ,$$

which shows M to be a space of constant curvature, provided that $n > 2$.

When $n = 2p$, from (3.6) it follows that

$$k = \frac{R}{2(n-1)}\ ,$$

and (3.5) becomes

$$(n-2)R_{\lambda\mu\nu\omega} + R_{\lambda\nu}g_{\mu\omega} - R_{\lambda\omega}g_{\mu\nu} + R_{\mu\omega}g_{\lambda\nu} - R_{\mu\nu}g_{\lambda\omega}$$
$$- \frac{R}{n-1}(g_{\lambda\nu}g_{\mu\omega} - g_{\lambda\omega}g_{\mu\nu}) = 0\ ,$$

which shows M to be conformally flat, provided that $n > 3$.

Thus we have the following theorem including the trivial cases where $p = 1$

and $n-1$; the converse part is proved by making use of $F_p(\pi_e)$.

Theorem. *In an n-dimensional Riemannian space M, if the mean curvature for p-plane is independent of the p-plane at each point, then*

(i) *M is an Einstein space, for* $p=1, n-1$ *and* $n>2$,

(ii) *M is of constant curvature, for* $n-1>p>1$ *and* $2p \neq n$,

(iii) *M is conformally flat, for* $n-1>p>1$ *and* $2p=n$.

The converse is also true.

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